Critical Loading of Struts and Structures.

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The main points of practical importance in the following paper are:—

(*a*) Discussion of stability of a prismatic homogeneous strut on simple supports, loaded laterally and longitudinally—criterion for failure.

(*b*) Equation of three moments generalised to include the case where the central axis is constrained to pass through $n+1$ non-collinear points, and where lateral loading and longitudinal end thrusts exist.

(*c*) Application to the case of n bays, and the calculation of the moments, reactions, etc.

(*d*) Case where one or more bays of Euler's critical length are present in the system. It is here shown that in general failure does not result.

(*e*) Failure will not in general take place if $n-1$ bays are of Euler's critical length, and the remaining one less.

(*f*) Case where one or more bays are twice Euler's critical length.

(*g*) General condition of failure for a structure of n bays.

(*h*) Conditions of strengthening or weakening by the addition of an extra bay to a structure critically loaded.

(*k*) Extension of the previous work to include the external loading due to wires in tension attached to the beam in the region of the supports.

(*l*) Clamped supports:—Determination of bending moments, etc., and condition of failure.

(*m*) Strengthening effect of clamped supports.

Failure in the above is understood to imply instability of geometrical configuration of the structure, under the critical loading.

§ 1. Many interesting questions of considerable practical importance relating to the failure of a structure under compression have been brought to light by a study of the problems of the strength of aeroplane frameworks. Unfortunately, however, very little advance has been made in this question on the theoretical side in its direct application to aeronautics, although a large number of investigations exist dealing with the general theory of elastic stability. It is extremely difficult to translate such results to make them directly useful in dealing with aeroplane frameworks, and, in fact, the general theory as ordinarily presented does not appear to

contain within it certain particular forms of failure that arise in this connection.

The present paper is an attempt to solve, as far as possible, problems relating to the strength of such a construction as a beam under end thrusts, and supported at intermediate points.

In a sense there are two types of failure of a structure. On the one hand, the material of a member may rupture on account of the yield stress having been exceeded during the course of the loading, while, on the other, the loading may be such that the geometrical configuration previously existing can no longer be maintained, even approximately. It is the second type of failure to which attention in this paper will be chiefly directed.

§ 2. The ground will be cleared initially by a simple discussion of an elementary case that may arise, such as, for example, when a prismatic, homogeneous, rectilinear uniform strut with simple supports at the ends is loaded both laterally and longitudinally.

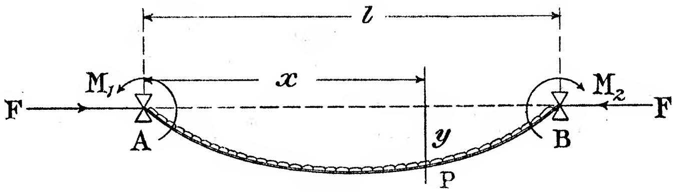

Fig. 1.

Let AB be the strut, simply supported at the ends A and B, and let w lb. per unit length be the intensity of lateral loading at P, of co-ordinates x and y. Let M_1 and M_2 be externally applied bending moments at the ends. Taking the origin at A, and AB as the axis of x, consider the equilibrium of the beam. The bending moment at P is given by

$$F \cdot y - M_2 + (M_2 - M_1)(l - x)/l + H(x),$$

where $H(x)$ is the bending moment at P due to the lateral loading alone. Equating this to the resisting moment, viz., $-EI$ (curvature) and assuming the deflections small, we find

$$-EI\frac{d^2y}{dx^2} = F \cdot y - M_1 + (M_1 - M_2)x/l + H(x), \tag{1}$$

where $H(x)$ does not involve M_1 and M_2.

This equation may be written

$$\frac{d^2y}{dx^2} + \lambda^2 y = M_1\lambda^2/F + x\lambda^2(M_2 - M_1)/Fl - \lambda^2 H(x)/F, \tag{2}$$

where $$\lambda^2 = F/EI. \tag{3}$$

Integrating, we obtain

$$y = C_1 \cos \lambda x + D_1 \sin \lambda x + M_1/F + x(M_2 - M_1)/Fl - \frac{\lambda^2}{F} \cdot \frac{H(x)}{D^2+\lambda^2}, \tag{4}$$

where the operator $1/(D^2+\lambda^2)$ has its usual significance, and the expression $-\lambda^2/F \,.\, H(x)/(D^2+\lambda^2)$ is easily evaluated in general by expanding $H(x)$ in a Fourier series. For convenience, represent this as a known function, $X(x)$, independent of M_1 and M_2. In the two special cases, firstly, in which there is no lateral loading, and, secondly, in which this loading is uniform, $X(x)$ becomes zero and $W/\lambda^2 F$ respectively.

To determine C_1 and D_1 we note that

$$y = 0,\ x = 0,$$

$$y = 0,\ x = l;$$

therefrom from (4)

$$0 = C_1 + M_1/F + X(0),$$

and

$$0 = C_1 \cos \lambda l + D_1 \sin \lambda l + M_2/F + X(l).$$

Hence

$$C_1 = -M_1/F - X(0). \tag{5}$$

$$D_1 = (M_1 \cos \lambda l - M_2)/F \sin \lambda l + [X(0) \cos \lambda l - X(l)]/\sin \lambda l. \tag{6}$$

The bending moment at any point is

$$\text{B.M.} = -EI\frac{d^2y}{dx^2} = \left[\left\{\frac{M_1 \cos \lambda l - M_2}{\sin \lambda l}\right\} + \left\{\frac{FX(0)\cos \lambda l - FX(l)}{\sin \lambda l}\right\}\right] \sin \lambda x$$
$$-\{M_1 + FX(0)\} \cos \lambda x + H(x) + FX(x). \tag{7}$$

Since all the terms on the right-hand side of this equation in general must be finite, with the exception of that involving $\sin \lambda l$, the bending moment at any point can only become infinite when

$$\sin \lambda l = 0, \tag{8}$$

unless at the same time

$$M_1 \cos \lambda l - M_2 + F\{X(0) \cos \lambda l - X(l)\} \longrightarrow 0. \tag{9}$$

Equation (8) indicates that when the length of the strut is given by

$$\lambda l = \pi,$$

i.e.

$$l = \pi\sqrt{(EI/F)}, \tag{10}$$

bending moments and deflections become excessively large compared with those allowable by the analysis. Equation (10) is the commonly recognised expression for "Euler's strut."

It should be particularly noted that under these conditions equation (9), reducing to

$$M_1 + M_2 = -F\{X(0) + X(l)\}, \tag{11}$$

furnishes a relation between the externally applied bending moments which may prevent the strut failing in the Eulerian sense, but this will be returned to later. For the moment it suffices to suggest that, in a structure in which M_1 and M_2 are determined by the loading and configuration of the system, it may be possible to have present a strut of Euler's length without failure resulting. Such cases will shortly be discussed.

§ 3. It appears, then, that in the simple case here treated, a critical loading of the beam is essentially associated with the production, at all points, of bending moments and deflections excessively large in comparison with those allowable by the assumptions upon which the analysis is based. This immediately suggests that failure in a corresponding sense may take place in a more complicated structure, and that these criticals will be furnished from the expressions determining the bending moments, on the assumption of small deflection, by inserting the condition that the former become infinite.

In the case of the simple strut, as treated by Euler, it has been shown that, up to the first critical loading given by

$$F = \pi EI/l^2,$$

the straight position of equilibrium is stable to small disturbances, but that, for loadings above this, the system deflects from this position to a new position of equilibrium, where the axis takes up the form of an elastica, the geometry of the structure being unstable. It will be assumed in the present discussion that, in the more complicated structures here to be treated, the lowest critical position furnished by the criterion already stated is one of instability with respect to the geometrical configuration, that, in fact, a loading greater than that stated here as critical will involve such deflections as would destroy the shape of the structure.

§ 4. *Case of a Uniform Beam under End Thrusts where the Central Axis is constrained to pass through $n+1$ Fixed Points.*—Let 0, 1, 2, 3, ..., n, be the $n+1$ supports whose positions are given, and let their distances from the datum line $0n$ be small compared with the lengths of the n bays 01, 12, 23,

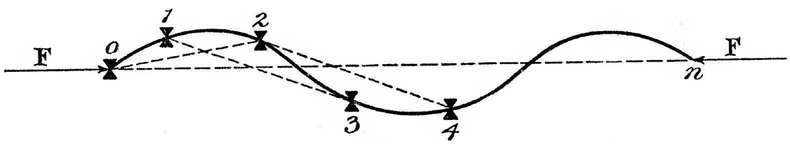

Fig. 2.

etc., and suppose E_1I_1, E_2I_2, ..., E_nI_n, are the flexural rigidities of the corresponding bays 01, 12, ..., $(n-1)n$. Let δ_1, δ_2, δ_3, etc., be the distances of 1, 2, 3, etc., from 02, 13, 24, etc., then δ_1, δ_2, δ_3, etc., will be termed the

deflections at the supports. It is clear that, as long as the beam is not otherwise constrained by wires inclined to the datum line On, at any intermediate points the longitudinal force F will be constant and equal to F along each bay as far as quantities of the second order. Modifications resulting from the attachment of such wires will receive special treatment later. It is proposed, in the first place, to find an expression for the bending moment at each point of the beam.

§ 5. Consider the rth and $(r+1)$th bays, ABC (fig. 3), laterally loaded according to some given law, and of lengths l_r and l_{r+1} respectively. The

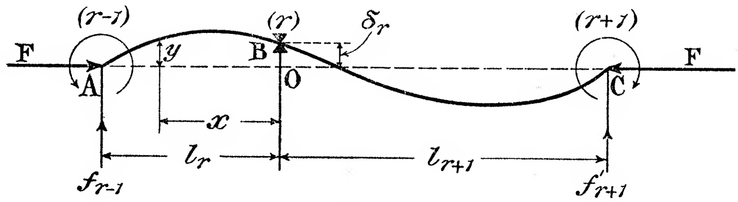

FIG. 3.

remainder of the beam to the left of the $(r-1)$th support, together with the support itself, may be replaced, since the deflections are supposed small, by forces F and f_{r-1}, parallel and perpendicular respectively to AC, and a bending moment M_{r-1}. Similarly, the beam to the right of C and the $(r+1)$th support itself may be replaced by F, f'_{r+1}, and M_{r+1}.

Taking the origin of co-ordinates at O, the bending moment at any point P (x, y) is

$$M_{r-1} - f_{r-1}(l_r - x) + F \,.\, y + \psi_r(x),\text{*}$$

where $\psi_r(x)$ is the moment about P of the lateral loading. Equating this to the restoring moment $-E_r I_r \, d^2y/dx^2$, y being small, it is found on investigation that

$$y = C_r \cos \lambda_r x + D_r \sin \lambda_r x - \{M_{r-1} - f_{r-1}(l_r - x)\}/F + \chi_r(x), \tag{12}$$

where
$$\lambda_r = \sqrt{(F/E_r I_r)} \tag{13}$$

and
$$\chi_r = -\frac{\lambda_r^2}{F} \cdot \frac{\psi_r(x)}{D^2 + \lambda_r^2}$$

similar to the simple case already treated. C_r and D_r are to be obtained from the end conditions, viz.,

$$x = 0, \qquad y = \delta_r,$$
$$x = l_r, \qquad y = 0.$$

* $\psi_r(x)$ is now the moment of the lateral force only on the portion of AB to the left of P, about the point P {*cf.* equation (1)}.

In addition it is to be remembered that

$$\begin{aligned} M_r &= \text{bending moment at B} \\ &= M_{r-1}+F\delta_r - f_{r-1}l_r + \psi_r(0). \end{aligned} \tag{14}$$

Accordingly

$$C_r = M_r/F - \{\psi_r(0)+F\chi_r(0)\}/F, \tag{15}$$

$$D_r = \{(M_{r-1}-M_r\cos\lambda_r l_r)+\cos\lambda_r l_r\,(\psi_r(0)+F\chi_r(0)\}/F\sin\lambda_r l_r - F\chi_r(l_r). \tag{16}$$

If θ_r be the inclination of the strut at the rth support to AC, then

$$\begin{aligned} \tan\theta_r &= \left(\frac{dy}{dx}\right)_0 \\ &= \lambda_r D_r + \chi_r'(0) - \{M_{r-1}-M_r+F\delta_r+\psi_r(0)\}/Fl_r, \\ &= A_r M_{r-1} + B_r M_r - \delta_r/l_r + K_r, \end{aligned} \tag{17}$$

where

$$A_r = \frac{l_r}{E_r I_r}\cdot\frac{1}{\lambda_r^2 l_r^2}\cdot\left\{\frac{\lambda_r l_r}{\sin\lambda_r l_r}-1\right\} = \frac{l_r}{E_k I_r}\cdot\alpha_r, \text{ say}. \tag{18}$$

$$B_r = \frac{l_r}{E_r I_r}\cdot\frac{1}{\lambda_r^2 l_r^2}\cdot\left\{1-\frac{\lambda_r l_r}{\tan\lambda_r l_r}\right\} = \frac{l_r}{E_r I_r}\cdot\beta_r, \text{ say}, \tag{19}$$

$$K_r = -B_r\psi_r(0)+\chi_r'(0)+\frac{\lambda_r}{\sin\lambda_r l_r}\{\chi_r(0)\cos\lambda_r l_r - \chi_r(l_r)\}. \tag{20}$$

In the same way, for the portion BC between the rth and $(r+1)$th supports, still maintaining the origin at O, similar equations are easily derived, giving as the gradient of the beem at the rth support

$$-\tan\theta_r = A_{r+1}M_{r+1}+B_{r+1}M_r-\delta_r/l_{r+1}+K_{r+1}'.* \tag{21}$$

This is easily obtained from equation (17) by remembering that the $(r+1)$th support now replaces the $(r-1)$th, the rth support remaining the same, and the $(r+1)$th bay takes the place of the rth.

By addition of equations (17) and (21) we find

$$A_r M_{r-1}+(B_r+B_{r+1})M_r+A_{r+1}M_{r+1} = \delta_r(1/l_r+1/l_{r+1})-(K_r+K_{r+1}'). \tag{22}$$

A_r, B_r, etc., are defined in terms of the dimensions of the bays, their flexural rigidities, and the longitudinal loading as shown by equations (18) and (19), and K_r and K_{r+1}' in addition by the lateral loading. This is the extended form for the equation of three moments, or Clapeyron's theorem when the axis of the beam is constrained to pass through given points, and when both

* The dashes on the K's indicate that these refer to the lateral load on the bay to the right of the point considered, so that K_r' has the same form as K_r, but the appropriate ψ's, etc., must be used. If the loading in a bay is uniform, $K_r = K_r'$ for that bay.

lateral and longitudinal loading exist. A slightly more general form of the equation will shortly be obtained to include the case where external loading of the nature of wires in tension and inclined to the axis exists.

§ 6. If there is no lateral loading, $K_r = 0$. If the beam be uniformly loaded with intensity w lb. per unit length, then it can easily be verified that

$$K_r = \frac{w_r l_r^3}{2E_rI_r} \cdot \frac{1}{\lambda_r^2 l_r^2} \cdot \left\{1 - \frac{2}{\lambda_r l_r} \tan \frac{\lambda_r l_r}{2}\right\} = -\frac{w_r l_r^3}{2} \cdot \frac{l_r}{E_rI_t} \cdot \gamma_r, \text{ say.} \quad (23)$$

When there are no end thrusts, $F = 0$, and consequently $\alpha \longrightarrow 1/6$, $\beta \longrightarrow 1/3$, $\gamma \longrightarrow 1/12$, and equation (22) becomes a simple extension of Clapeyron's theorem for the case where the supports are not collinear.

The non-dimensional functions α, β, and γ will be found very convenient for actual calculation in practical applications, and they are easily plotted for variations of the non-dimensional expression ϕ where

$$\phi = \lambda l = l\sqrt{(F/EI)}.$$

Equation (22) can now be applied directly to the solution of the problem under discussion where there are $n+1$ supports.

§ 7. *Case of* n *Bays.*—Let there be $n+1$ supports, figured 0, 1, 2, ..., n, and spaced as in the diagram, so that the rth bay appears to the left of the support numbered r. Let the bending moments at these supports be

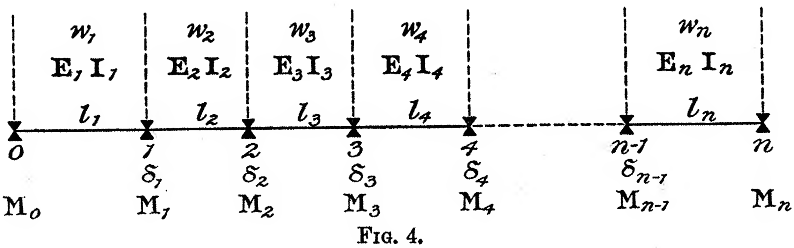

Fig. 4.

$M_0, M_1, \dots, M_n$, the deflection as already defined $\delta_1, \delta_2, \dots, \delta_{n-1}$, and suppose the lateral loading considered uniform along each bay, $l_1, l_2, \dots, l_n$, is $w_1, w_2, \dots, w_n$. $E_1I_1, E_2I_2, \dots, E_nI_n$, are as before the flexural rigidities of the various bays.

Applying the generalised form of the equation of three moments derived above, (22), to the three supports of every pair of contiguous bays, we obtain the following system of equations

Central support of trio.

1 $$M_0A_1+M_1(B_1+B_2)+M_2A_2 = \delta_1(1/l_1+1/l_2)-(K_1+K_2), \tag{24$_1$}$$

2 $$M_1A_2+M_2(B_2+B_3)+M_3A_3 = \delta_2(1/l_2+1/l_3)-(K_2+K_3), \tag{24$_2$}$$

....

r $$M_{r-1}A_r+M_r(B_r+B_{r+1})+M_{r+1}A_{r+1} = \delta_r(1/l_r+1/l_{r+1})-(K_r+K_{r+1}), \tag{24$_r$}$$

....

$n-1$ $$M_{n-2}A_{n-1}+M_{n-1}(B_{n-1}+B_n)+M_nA_n = \delta_{n-1}(1/l_{n-1}+1/l_n)-(K_{n-1}+K_n), \tag{24$_{n-1}$}$$

where A_r, B_r, and K_r have the significance already attached to them in equations (18), (19), and (20).

These constitute $n-1$ equations from which, if the bending moments at any two supports, usually 0 and n, be known, the remaining bending moments in general are uniquely determined. The equations are fortunately in a simple form for rapid solution, in sharp contrast to the method of strain energy, where generally all the variables appear in all the equations. Once the bending moments at all the supports have been evaluated, the determination of the bending moment at any point in each bay is a simple matter.

From equation (12), by differentiating twice, it is easily found that the bending moment at any point in the bay to the left of the rth support at distance x from that support is given by

$$\begin{aligned} \text{B.M.} &= -E_rI_r\frac{d^2y}{dx^2} \\ &= E_rI_r\lambda_r^2\{C_r\cos\lambda_rx+D_r\sin\lambda_rx\}-E_rI_r\chi_r''(x) \\ &= F\{C_r\cos\lambda_rx+D_r\sin\lambda_rx\}-\{\psi_r(x)-F\chi_r(x)\}, \end{aligned} \tag{25}$$

where C_r and D_r are furnished by equations (15) and (16).

For this case where each bay is uniformly loaded

$$\psi_r(x) = \tfrac{1}{2}w_r\,.\,(l_r-x)^2, \tag{26}$$

therefore
$$\begin{aligned} \chi_r(x) &= -\frac{\lambda_r^2}{F}\cdot\frac{\psi_r(x)}{D^2+\lambda_r^2} \\ &= -\psi_r(x)/F+w_r/\lambda_rF, \end{aligned} \tag{27}$$

therefore
$$\psi_r(x)-F\chi_r(x) = w_r/\lambda_r^2, \tag{28}$$

therefore
$$C_r = M_r/F-w_r/\lambda_rF, \tag{29}$$

$$D_r = \{M_{r-1}-M_r\cos\lambda_rl_r-(1+\cos\lambda_rl_r)\,w_r/\lambda_r^2\}/F\sin\lambda_rl_r. \tag{30}$$

The reactions at the supports are likewise easily evaluated, as follow:—

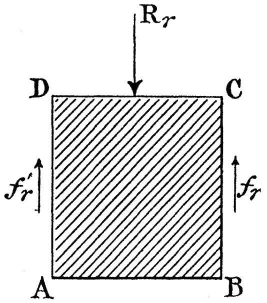

FIG. 5.

Consider a portion ABCD of the beam, enclosing the rth support, and let the reaction of the support upon it be R, acting downwards, while f_r and f_r' are the shearing forces to the right and left respectively of the support, then

$$R_r = f_r + f_r'.$$

From equation (14)

$$f_r = M_r - M_{r+1} + F\delta_{r+1} + w_{r+1} l_{r+1}^2/2.$$

By taking moments about the $(r-1)$th support for forces to the right of that point we find

$$f_r' = M_r - M_{r-1} + F\delta_{r-1} + w_r l_r/2.$$

From these R_r is at once derived. Using the notation given in fig. 4, and measuring δ as in fig. 3, the bending moments when positive will tend to make the beam concave on the under surface, and the reaction of the support upon the beam will be positive when directed downwards.

§ 8. In general, equations (24_1), (24_2), ..., (24_{n-1}), will suffice to determine the values of the bending moments at the supports, and consequently also at any point of the beam, according to the method outlined above. It has already been seen that the type of critical failure here investigated is associated with the production of infinite bending moments, but before proceeding to the general treatment of this question, it will be worth while to enquire into the suggestion already thrown out, that the presence of a bay of Euler's critical length would not of itself involve failure of the type here considered. In common engineering practice failure is often assumed to be the case.

§ 9. *Case where One or More of the Bays are of Euler's Critical Length.*—

Suppose $$\lambda_r l_r = \pi,$$

i.e. $$F = \pi^2 E_r I_r / l_r^2.$$

Such a bay, unassociated with the remainder of the structure, would

collapse under the load. It is proposed to enquire whether or not the presence of the remainder of the structure can exert a strengthening effect. The expression for the shape of the rth bay is given by equation (12) in the form

$$y = C_r \cos \lambda_r x + D_r \sin \lambda_r x + \chi_r(x) - \{M_{r-1} - f_{r-1}(l_r - x)\}/F, \tag{12}$$

where f_{r-1} satisfies equation (14), viz.,

$$f_{r-1} = (M_{r-1} - M_r)/l_r + F\delta_r/l_r + \psi_r(0)/l_r. \tag{14}$$

Remembering that

$$x = 0, \qquad y = \delta_r,$$

$$x = l_r, \qquad y = 0,$$

and

$$\lambda_r l_r = \pi,$$

and inserting these in (12) modified by (14) we find

$$0 = C_r - M_r/F + \psi_r(0)/F + \chi_r(0),$$

$$0 = -C_r - M_{r-1}/F + \chi_r(l_r).$$

Therefore

$$C_r = -M_{r-1}/F + \chi_r(l_r) \tag{31}$$

and

$$M_r + M_{r-1} = \psi_r(0) + F\{\chi_r(0) + \chi_r(l_r)\}. \tag{32}$$

For no lateral load (32) takes the simplified form

$$M_r + M_{r-1} = 0, \tag{33}$$

and for uniform load

$$M_r + M_{r-1} = 2w_r/\lambda_r^2 = 2w_r l_r^2/\pi^2. \tag{34}$$

The constant D_r can at once be determined in terms of the M's by the statement that the gradient at the $(r-1)$th support for the rth bay is equal to the gradient at the same point for the $(r-1)$th bay, the expression for the latter involving no indeterminateness if it is not itself of Euler's length. If, however, the $(r-1)$th bay is also of Euler's length, then that bay must be treated in the same way as the rth above, with reference to the $(r-2)$th. It follows that all the coefficients are determinate, provided that one bay is not of Euler's length. Corresponding with each Euler's bay, there will, of course, exist a condition of the form given in (34), and under these circumstances the structure will not fail provided it can be shown that none of the bending moments become infinite for any value of the longitudinal load F up to that here considered. This will come out from a discussion of the equations for bending moments.

It will be necessary to investigate the limiting forms which the coefficients A_r and B_r assume in this case. It is clear, of course, that both these quantities become in the limit infinitely great, but

$$\underset{\lambda_r l_r = \pi}{\text{Lim}} (A_r - B_r) = \underset{\lambda_r l_r = \pi}{\text{Lim}} \left\{ \frac{l_r}{E_r I_r} \cdot \frac{1}{\lambda_r^2 l_r^2} \left(\frac{\lambda_r l_r}{\sin \lambda_r l_r} + \frac{\lambda_r l_r}{\tan \lambda_r l_r} - 2 \right) \right\}$$

$$= \underset{\lambda_r l_r = \pi}{\text{Lim}} \left\{ \frac{l_r}{E_r I_r} \left(\frac{1 + \cos \lambda_r l_r}{\sin \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\} \frac{1}{\lambda_r l_r}$$

$$= \underset{\lambda_r l_r = \pi}{\text{Lim}} \left\{ \frac{l_r}{E_r I_r} \cdot \frac{1}{\lambda_r l_r} \left(\frac{1}{\tan \frac{1}{2} \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\}$$

$$= -\frac{2 l_r}{\pi^2 E_r I_r}. \tag{35}$$

Equations (24_1) to (24_{n-1}) determine the bending moments, but (24_{r-1}) and (24_r) require special treatment on account of the limiting forms of A_r, B_r, and K_r. Setting out these two equations we obtain

$$M_{r-2}A_{r-1} + M_{r-1}B_{r-1} + (M_{r-1}B_r + M_r A_r + K_r)$$
$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - K_{r-1}, \tag{36}$$

$$(M_{r-1}A_r + M_r B_r + K_r) + M_r B_{r+1} + M_{r+1}A_{r+1} = \delta_r (1/l_r + 1/l_{r+1}) - K_{r+1}. \tag{37}$$

The groups of terms enclosed in the brackets to the left-hand side of these equations clearly require modification. The following relations have been found to hold for uniform loading:—

$$\text{Lim}\,(M_{r-1} + M_r) = \frac{2w_r}{\lambda_r^2} \tag{38}$$

and

$$\text{Lim}\,(A_r - B_r) = -\frac{2l_r}{\pi^2 E_r I_r}. \tag{39}$$

Subtracting (36) from (37) we find

$$(M_{r-2}A_{r-1} + M_{r-1}B_{r-1}) - (M_r B_{r+1} + M_{r+1}A_{r+1}) + (M_{r-1} - M_r)(B_r - A_r)$$
$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - \delta_r (1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1}.$$

The only term which adopts an indeterminate form is

$$\underset{\lambda_r l_r = \pi}{\text{Lim}} (M_{r-1} - M_r)(B_r - A_r) \rightarrow \frac{2l_r}{\pi^2 E_r I_r}(M_{r-1} - M_r).$$

Equations (24_{r-1}) and (24_r) must now be replaced by

$$M_{r-1} + M_r = \frac{2w_r l_r^2}{\pi^2}, \tag{40}$$

and $$M_{r-2}A_{r-1} + M_{r-1}\left(B_{r-1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_r \left(B_{r+1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_{r+1}A_{r+1}$$
$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - \delta_r (1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1}. \tag{41}$$

The two equations (24_{r-1}) and (24_r), containing coefficients which become infinite or indeterminate, can now, therefore, be replaced by (40) and (41), in which all the bending moments occurring in the previous two are still present, but the coefficients are all finite and known. It follows that in general the

system of equations (24_1), (24_2), ..., (24_{r-2}), (40), (41), ..., (24_{n-1}), will determine finite bending moments for all the supports, and, consequently, also for every point in the beam. Only under special conditions about to be treated will failure occur due to buckling.

Summing up, then, a beam supported at any number of simple supports will not fail, in general, through the bending moments becoming excessive, even if some of the bays are of Euler's critical length, provided at least one bay is not of this length. Whether or not this will correspond with a stable loading of the structure will depend on whether or not the actual longitudinal load, if any, that would produce infinite bending moments is greater or less than Euler's load for the bays in question. It appears then that the presence of bays of Euler's length is not in itself either a necessary or sufficient criterion for instability. An accurate investigation of the conditions under which failure will take place will be given and a comparison between the crippling load, thus determined, and Euler's load will decide whether instability exists at the latter position.

§ 10. *Cases where One of the Bays is twice Euler's Length.*—The equation for the rth bay is

$$y = C_r \cos \lambda_r x + D_r \sin \lambda_r x - \{M_{r-1} - f_{r-1}(l_r - x)\}/F - \chi_r(x), \tag{12}$$

when $x = 0, \quad y = \delta_r,$

$$x = l_r, \quad y = 0,$$

and, remembering that $2\pi = \phi_r = \lambda_r l_r$, we find

$$\delta_r = C_r - (M_{r-1} - f_{r-1} l_r)/F + \chi_r(0),$$

$$0 = C_r - (M_{r-1})/F + \chi_r(l_r),$$

therefore
$$\delta_r = \frac{f_{r-1} l_r}{F} + \delta_r(0) - \chi_r(l_r).$$

Now
$$f_{r-1} l_r/F = \delta_r + (M_{r-1} - M_r)/F + \chi_r(0)/F, \tag{14}$$

therefore
$$M_{r-1} - M_r = F\{\chi_r(l_r) - \chi_r(0)\} - \psi_r(0). \tag{42}$$

This is the modified equation for this case corresponding with that given in equation (32).

Proceeding by the same method adopted there the equations that ought to furnish the bending moments for the rth bay are

$$M_{r-2}A_{r-1} + M_{r-1}(B_{r-1} + B_r) + M_r A_r + K_r = \left(\frac{1}{l_{r-1}} + \frac{1}{l_r}\right)\delta_{r-1} - K_{r-1}, \tag{43}$$

$$M_{r+1}A_{r+1} + M_{r-1}A_r + M_r(B_r + B_{r+1}) + K_r = \delta_r\left(\frac{1}{l_r} + \frac{1}{l_{r-1}}\right) - K_{r+1}. \tag{44}$$

In these equations, however, when $\phi_r = 2\pi$, A_r and B_r become infinite. It remains to show that the bending moments under these circumstances must also become infinite. From the last two equations, by subtraction, we find

$$M_{r+1}A_{r+1} + M_r(B_r + B_{r+1} - A_r) + M_{r-1}(A_r - B_{r-1}) - M_{r-2}A_{r-1}$$
$$= \delta_r\left(\frac{1}{l_r} + \frac{1}{l_{r-1}}\right) - K_{r+1} - \delta_{r-1}\left(\frac{1}{l_{r-1}} + \frac{1}{l_r}\right) + K_{r-1}, \quad (45)$$

therefore

$$(M_r - M_{r-1})(B_r - A_r) - M_{r-2}A_{r-1} - M_{r-1}B_{r-1} + M_rB_{r+1} + M_{r+1}A_{r+1} = \text{etc.} \quad (46)$$

Now $M_r - M_{r-1}$ is already found in terms of the loading and in general is not zero, while

$$\lim_{\phi_r = 2\pi} (B_r - A_r) \rightarrow \infty .$$

Hence this equation, giving a relation between the bending moments M_{r-2}, M_{r-2}, M_r, and M_{r+1}, contains an infinite term, and since the coefficients of these bending moments and the remaining terms are all finite, it follows that one at least of the bending moments must become infinite, and the structure must fail. This is, however, not the lowest critical loading.

§ 11. *Conditions of Failure of a Supported System.*—The criterion of failure which will be utilised in the present discussion is, as already explained, the production of infinite binding moments at the supports. Associated with this will be the assumption already known to be valid in the case of a single strut, that for values of the longitudinal force greater than the least that will produce these infinite bending moments, the geometry of the structure will not be maintained, the failure, in this sense, thus corresponding with instability. It remains, therefore, to write down the mathematical expression which must be satisfied when the bending moments are infinite and to determine the least root, regarding it as an equation in F, the longitudinal force.

§ 12. *Condition of Failure for the Case of n Bays, i.e., n + 1 Supports* (fig. 4).—The equations (24_1), …, (24_{n-1}) are sufficient in general to determine the bending moments at the supports. These equations being linear, each of the bending moments may be expressed as the ratio between two determinants whose constituents are functions of the lengths of the bays, deflections, and the lateral and longitudinal loading. The bending moments will become infinite when, and only when, the denominator vanishes. This condition, assuming the bending moments at the end supports zero, expressed in determinantal form is:—

$$0 = \begin{vmatrix} B_1+B_2, & A_2, & 0, & 0, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ A_2, & B_2+B_3, & A_3, & 0, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & A_3, & B_3+B_4, & A_4, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & 0, & A_4, & B_4+B_5, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ 0, & 0, & 0, & A_5, \ldots\ldots & \ldots\ldots\ldots\ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & A_{n-3}, & 0, & 0 \\ 0, & 0, & 0, \ldots\ldots & A_{n-3}, & B_{n-3}+B_{n-2}, & A_{n-2}, & 0 \\ 0, & 0, & 0, \ldots\ldots & 0, & A_{n-2}, & B_{n-2}+B_{n-1}, & 0 \\ 0, & 0, & 0, \ldots\ldots & 0, & 0, & A_{n-1}, & B_{n-1}+B_n \end{vmatrix} \tag{47}$$

since $\qquad M_1 = 0, \qquad M_{n+1} = 0.$

$$A_n = \frac{l_n}{E_nI_n} \cdot \frac{1}{\phi_n^2}\left(\frac{\phi_n}{\sin\phi_n} - 1\right), \qquad B_n = \frac{l_n}{E_nI_n} \cdot \frac{1}{\phi_n^2}\left(1 - \frac{\phi_n}{\tan\phi_n}\right) \tag{48}$$

where $\qquad \phi_n = \lambda_n l_n.$

§ 13. For any particular case this symmetrical determinant can be very easily evaluated by the following scheme.

Writing the value of this determinant for the case bays of Nos. 1, 2, etc., as Δ_1, Δ_2, etc., we obtain—

		No. of bays.
1	$= \Delta_1$	1
$(B_1+B_2)\,\Delta_1$	$= \Delta_2$	2
$(B_2+B_3)\,\Delta_2 - A_2^2\Delta_1$	$= \Delta_3$	3
$(B_3+B_4)\,\Delta_3 - A_3^2\Delta_2$	$= \Delta_4$	4
$(B_4+B_5)\,\Delta_4 - A_4^2\Delta_3$	$= \Delta_5$	5
....	....	
$(B_{n-1}+B_n)\,\Delta_{n-1} - A_{n-1}^2\Delta_{n-2}$	$= \Delta_n$	n

(49)

When the value of ϕ for any bay reaches π, A and B become infinite and accordingly the determinant in general also becomes infinite. To avoid this it would be convenient to divide throughout by the product of the A's, so that instead of taking $\Delta_n = 0$ as the determinantal equation the condition

$$D_n = \frac{1}{A_1 \,.\, A_2 \,.\, A_3 \ldots A_n \,.}\,\Delta_n \tag{50}$$

may be used.

Two special cases need comment corresponding with any one of the A's becoming either zero or infinite. In the former this can only happen when l is made indefinitely small, but, as can be seen, in neither case is any real difficulty involved in the calculations. It is interesting to note that $D_1 = 1/A_1 = 0$, the criterion of failure for one bay, gives the result obtained by Euler.

§ 14. The determinantal condition (53) involves $\phi_1, \phi_2, \phi_3, \ldots$, which are all known functions of F, the end thrust, and is independent of the lateral loading. Thus

$$\phi_r^2 = l_r^2 F/E_r I_r,$$

and a solution of the problem is obtained immediately the smallest value of F satisfying the condition (53) is found. It has been assumed that this longitudinal thrust is practically constant along the whole length of the beam, since the reactions at the joints acting perpendicular to the central axis can only modify F by second-order quantities. The analysis, however, can be applied more generally, with very slight modification, to the more practical case—as far as aeroplane frameworks are concerned—by supposing wires under tension fixed to the beam at the supports with a component force directed along the central axis. A simple resolution of forces suffices to show that the bays are now under end thrusts, not necessarily all equal. The rth bay may then be represented as being under a longitudinal force F_r, expressible of course in terms of the force in one particular. ϕ_1, ϕ_2, etc., now take the more generalised form

$$\phi_1 = l_1 \sqrt{\frac{F_1}{E_1 I_1}}, \qquad \phi_r = l_r \sqrt{\frac{F_r}{E_r I_r}}, \text{ etc.}$$

and the analysis proceeds as before. The determinantal condition (47), being an expression containing all the ϕ's, can then be easily expressed as a function of the ϕ for any particular bay. When the point of attachment of the wires is not on the neutral axis, a bending moment of known magnitude is introduced at the corresponding support, and the appropriate modification must then be inserted in the equations for three moments.

§ 15. *Clamped Supports.*—Suppose at any point in the beam the central axis besides being constrained to pass through a particular point is, by means of a clamp of some nature, compelled to have its central axis in a given direction at the point. Such a support will be termed a clamped support. It is proposed to consider what effect the introduction of such constraints has upon the strength of the structure by the determination of the positions of the criticals in comparison with those produced when the same support is unclamped. For the present purpose, a clamped support is in effect equivalent to two simple supports infinitely close together, and the discussion

consequently centres round the limiting forms of the conditions already obtained when one or more of the terms l_r tends to zero.

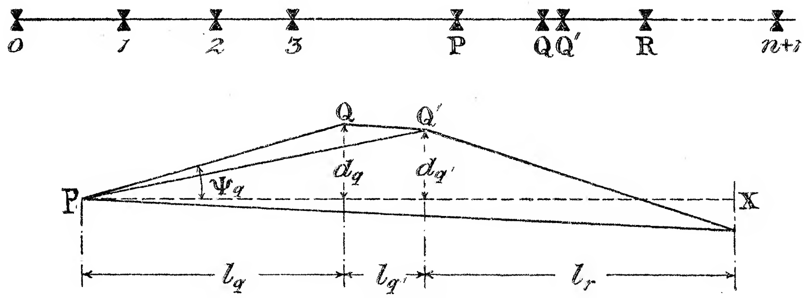

Fig. 6.

Let 0, 1, ..., $n+1$ be a beam supported at $n+1$ points by simple supports, and at QQ′ by two neighbouring supports, to constitute a clamped support. Consider a small section, PQQ′R, and let the heights of Q and Q′ above the datum line PX be d_q and d_q' respectively, then

$$\delta_q = \text{the distance of Q above PQ}'$$

$$= d_q - d_q' \frac{l_q}{l_q + l_q'} \text{ approx.}$$

$$= d_q - d_q'(1 + l_q'/l_q)^{-1} = d_q - d_q'(1 - l_q'/l_q).$$

Hence $\quad \delta(1/l_q + 1/l_q') = \{d_q - d_q'(1 - l_q'/l_q)\}(1/l_q + 1/l_q').$

In the limit, when Q and Q′ coincide,

$$\lim_{l_q' = 0} \delta_q'(1/l_q' + 1/l_r) = \theta_q - \Psi_r. \tag{51}$$

where Ψ_q is the inclination of PQ to the datum line and θ_q the inclination of the tangent to the same line, measuring angles in an anti-clockwise direction.

Similarly, δ_q' = distance of Q′ from QR, and hence

$$\lim_{l_q' = 0} \delta_q'(1/l_q' + 1/l_r) = \theta_q - \Psi_r. \tag{52}$$

In the light of these, the equations of three moments become modified by the substitution of the expression found in (56) and (57) for the deflection terms, and the analysis proceeds as before. It is evident, of course, both physically and mathematically, that the system of equations is separated at the points of double support into series of self-inclusive sets so that each portion of the beam between the clamped supports may be treated separately. It remains, therefore, to discuss only the cases where the beam is clamped at either or at both ends.

§ 16. *Beam Supported at $n-1$ Intermediate Points and Clamped at Both Ends.*

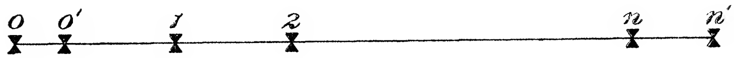

Fig. 7.

Let $0, 0', 1, \ldots, n', n$ be the beam supported simply at $1, 2, \ldots (n-1)$, and clamped at $0'0$ and $n'n$. Let $0'0$ and $n'n$ be l_0' and l_n' respectively, where these limit to zero. Then if δ_0' = deflection at $0'$ and δ_n = deflection at n,

$$\delta_0'\left(\frac{1}{l_0}+\frac{1}{l_1}\right) = \theta_0-\Psi_1$$

and

$$\delta_0'\left(\frac{1}{l_{n-1}'}+\frac{1}{l_n'}\right) = -\theta_n+\Psi_n.$$

Remembering that

$$A_0 = 0, \quad B_0 = 0, \quad A_n' = 0 \text{ and } B_n' = 0,$$

the system of equations for the moments at the supports takes the form

$$M_0B_1+M_1A_1 = \theta_0-\Psi_1-K_1,$$

$$M_0A_1+M_1(B_1+B_2)+M_2A_2 = \delta_1\left(\frac{1}{l_1}+\frac{1}{l_2}\right)-(K_1+K_2),$$

....

....

$$M_{n-2}A_{n-2}+M_{n-1}(B_{n-2}+B_{n-1})+M_nA_{n-1} = \delta_{n-1}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)-(K_{n-1}+K_n),$$

$$M_{n-1}A_{n-1}+M_nB_{n-1} = -\theta_n+\Psi_n-K_n. \tag{53}$$

These provide $n+1$ equations for the $n+1$ bending moments which now include M_0 and M_n as unknowns.

The determinantal condition corresponding with critical loading is of course

$$\begin{vmatrix} B_1, & A_1, & 0, & 0, & \ldots\ldots\ldots\ldots & 0 \\ A_1, & B_1+B_2, & A_2, & 0, & \ldots\ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & \ldots\ldots\ldots\ldots & & A_{n-2}, & B_{n-2}+B_{n-1}, & A_{n-1} \\ 0, & \ldots\ldots\ldots\ldots & & 0, & A_{n-1}, & B_{n-1} \end{vmatrix} = 0. \tag{54}$$

§ 17. The case where the double support is at one end only may likewise be simply treated.

The effect upon the strength of the structure of the introduction of the double support may, qualitatively at least, be seen from simple considerations

without reference to the determinant of failure. It has already been shown that if a structure of any number of simple supports be so spaced as to be critical under the loading, then the addition of one more bay at the end will have a strengthening effect, provided that the length of the bay is not greater than the length of Euler's strut. Remembering that the substitution of a clamped for a simple support is ultimately merely the addition of another bay of infinitely small length, it follows that clamping one or both ends is equivalent to a direct addition of strength. The exact determination of the magnitude of this increased safety is, of course, to be obtained by finding the smallest root of the determinant regarded as an equation in F. In fact, the whole problem of the most efficient disposition of the supports to provide the strongest structure, in the sense that the critical longitudinal force is a maximum among those for all possible dispositions, can be treated with comparative simplicity by a discussion of the determinants already set down.

The authors are indebted to Prof. Love for his helpful criticisms and valuable suggestions regarding the notation.

The Electromagnetic Inertia of the Lorentz Electron.

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1. In a recent paper* on "The Effective Inertia of Electrified Systems moving with High Speed," G. W. Walker extends his previous investigation to the case of a perfectly conducting oblate spheroid, whose axis lies in the direction of motion, for the time being, of its centre, and whose eccentricity is equal to k, where k denotes the ratio of the speed of the centre to that of light, also for the instant under consideration. He finds (Note 1) that the longitudinal and transverse masses are equal respectively to

$$2e^2(1-\tfrac{1}{5}k^2)/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2(1+\tfrac{1}{60}k^2)/3ac^2(1-k^3)^{1/2}, \qquad (1)$$

instead of
$$2e^2/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2/3ac^2(1-k^2)^{1/2}, \qquad (2)$$

the values given by the mass formulæ of Lorentz and required by the Principle of Relativity.

* 'Roy. Soc. Proc.,' A, vol. 93, p. 448 (1917).